

Some interactions between analysis and functor categories

J. Daisie Rock

KU Leuven, UGent

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Workshop on Categorification in Functional Analysis

The Plan

Stability Conditions

Nakayama Categories

Preprojective Categories

Pick your favourite algebraically closed field \mathbb{k} of characteristic 0.

We do not always need all of these assumptions but it simplifies this presentation.

The Plan

Stability Conditions

Nakayama Categories

Preprojective Categories

Stability Conditions

“Definition”

A **continuous quiver of type A** is a category that looks like this:



- ▶ We index the vertices over \mathbb{R} .
- ▶ The red parts “go down” and the blue parts “go up”.
- ▶ The open vertices are not part of the category.
- ▶ A purple vertex is a source if it has red on the left and blue on the right.
- ▶ A purple vertex is a sink if it has blue on the left and red on the right.
- ▶ In this presentation, we insist that we only have finitely-many sections of blue and red, alternating, and thus finitely-many purple vertices (possibly none).

Stability Conditions

Definition

- ▶ A **representation** M of a continuous quiver \mathcal{Q} of type A is a functor from \mathcal{Q} to \mathbb{k} -vector spaces.
- ▶ M is **pointwise finite-dimensional (pwf)** if M factors through finite-dimensional \mathbb{k} -vector spaces.
- ▶ It is an **interval module** if there is an interval $I \subset \mathbb{R}$ such that $M(x) = \mathbb{k}$ when $x \in I$ and $M(x) = 0$ otherwise. Linear maps are the identity where possible and 0 otherwise.

Definition

An object M in an abelian category is **indecomposable** if, whenever $M \cong M_1 \oplus M_2$, one of M_1 or M_2 is isomorphic to M and the other is the 0 object.

Stability Conditions

Facts

- ▶ The pwf representations of a continuous quiver \mathcal{Q} of type A , denoted $\text{rep}^{\text{pwf}}(\mathcal{Q})$, is a \mathbb{k} -linear abelian category (an abelian category enriched over finite-dimensional \mathbb{k} -vector spaces).
- ▶ An interval representation M of a continuous quiver of type A is indecomposable and $\text{End}(M) \cong \mathbb{k}$.

Theorem (Botnan–Crawley-Boevey, Igusa–R-Todorov)

Every pwf representation of a continuous quiver of type A decomposes uniquely up to isomorphism into a (possibly infinite) direct sum of interval representations.

Stability Conditions

Example

For simplicity, we will use this continuous quiver \mathbb{R} :



We have $\text{Ob}(\mathbb{R}) = \mathbb{R}$ and

$$\text{Hom}_{\mathbb{R}}(x, y) = \begin{cases} \{\alpha_{yx}\} & x \leq y \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\alpha_{xx} = \mathbf{1}_x$ and $\alpha_{zy}\alpha_{yx} = \alpha_{zx}$, when $x \leq y \leq z$.

An interval representation of \mathbb{R} is $M_{[0,10)}$:

$$M_{[0,10)}(x) = \begin{cases} \mathbb{k} & 0 \leq x < 10 \\ 0 & \text{otherwise.} \end{cases} \quad M_{[0,10)}(\alpha_{yx}) = \begin{cases} \mathbf{1}_{\mathbb{k}} & 0 \leq x \leq y < 10 \\ 0 & \text{otherwise.} \end{cases}$$

Stability Conditions

“Definition”

The subcategory $\text{rep}^{\text{fp}}(\mathbb{R})$ of $\text{rep}^{\text{pwf}}(\mathbb{R})$ is the abelian subcategory whose interval indecomposables are of the form $M_{[a,b]}$ for each $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, where $a < b$. The objects are finite direct sums of these indecomposables (and the 0 object).

Facts

- ▶ The representations $P_x := M_{[x,+\infty)}$ are projective in $\text{rep}^{\text{pwf}}(\mathbb{R})$.
- ▶ Every interval indecomposable $M_{[a,b]}$ in $\text{rep}^{\text{fp}}(\mathbb{R})$ has a projective resolution of the form

$$P_b \hookrightarrow P_a \twoheadrightarrow M_{[a,b]}.$$

(The “fp” stands for “finitely-presented”.)

Stability Conditions

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We assume all functions $f : U \rightarrow \overline{\mathbb{R}}$ in this section are integrable on their domain $U \subseteq \overline{\mathbb{R}}$.

Definition

The half- δ functions δ_x^- and δ_x^+ at $x \in \mathbb{R}$ have the following property for any $f : [a, b] \rightarrow \overline{\mathbb{R}}$ with $a < x < b$:

$$\int_a^x (f(t) + \delta_x^-) dt = \left(\int_a^x f(t) dt \right) + 1,$$

$$\int_x^b (f(t) + \delta_x^-) dt = \int_x^b f(t) dt.$$

$$\int_a^x (f(t) + \delta_x^+) dt = \int_a^x f(t) dt,$$

$$\int_x^b (f(t) + \delta_x^+) dt = \left(\int_x^b f(t) dt \right) + 1.$$

Stability Conditions

Definition

We say a function $f : \overline{\mathbb{R}} \rightarrow \mathbb{R}$ has **bounded variation** if

$$\left(\int_{-\infty}^{\infty} |f(t)| dt \right) \in \mathbb{R}.$$

Definition

For each $x \in \mathbb{R}$, we define $\Delta_x^-, \Delta_x^+ : \overline{\mathbb{R}} \rightarrow \mathbb{R}$ by

$$\Delta_x^-(z) = \int_{-\infty}^z \delta_x^- dt = \begin{cases} 0 & z < x \\ 1 & z \geq x \end{cases}$$
$$\Delta_x^+(z) = \int_{-\infty}^z \delta_x^+ dt = \begin{cases} 0 & z \leq x \\ 1 & z > x. \end{cases}$$

Stability Conditions

Definition

We also introduce the following functions:

$$\Delta_{-\infty}^+(z) = \begin{cases} 0 & z = -\infty \\ 1 & z > -\infty \end{cases} \quad \Delta_{+\infty}^-(z) = \begin{cases} 0 & z < +\infty \\ 1 & z = +\infty. \end{cases}$$

We can think of these like the other Δ functions as if we had half- δ functions at each infinity, on the correct side.

Definition

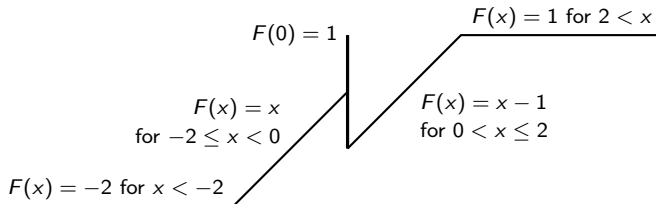
We say a function $F : \overline{\mathbb{R}} \rightarrow \mathbb{R}$ is **useful** if it satisfies the following.

1. $F = f + u_{-\infty}^+ \Delta_{-\infty}^+ + \sum_{x \in \mathbb{R}} (u_x^- \Delta_x^- + u_x^+ \Delta_x^+) + u_{+\infty}^- \Delta_{+\infty}^-$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function of bounded variation and all u_x^-, u_x^+ are in \mathbb{R} , for $x \in \overline{\mathbb{R}}$.
2. The sum $\sum_{x \in \mathbb{R}} (|u_x^-| + |u_x^+|)$ converges in \mathbb{R} .

Stability Conditions

Example

Consider $F(x) = f(x) + \Delta_0^-(x) - 2\Delta_0^+(x)$ when f is given by:
 $f(x) = x$ if $|x| \leq 2$, $f(x) = -2$ for $x < -2$, $f(x) = 2$ if $x > 2$,
 $u_0^- = 1$, $u_0^+ = -2$, and $u_x^\pm = 0$ for all other $x \in \mathbb{R} \cup \{\pm\infty\}$.



Even though $F(0) = 1$ we still draw the entire vertical line connecting the left limit, right limit, and function value. We need to draw useful functions this way for later.

Stability Conditions

Definition

Given a useful function F , we define $F_{\max}(x)$ to be the maximum among: the left limit at x , the right limit at x , the value $F(x)$.

We define $F_{\min}(x)$ to be the minimum of the same set.

For $x = \pm\infty$, we take the maximum or minimum over the two elements that make sense.

Definition

A **red-blue function pair** for \mathbb{R} is a pair of useful functions (R, B) satisfying the following.

1. For all $x \in \overline{\mathbb{R}}$ we have $B_{\min}(x) = B(x)$.
2. We have $u_{-\infty}^+ = 0$.
3. For all $x \in \overline{\mathbb{R}}$ we have $R(x) \leq B_{\min}(x)$.
4. We have $R(-\infty) = B(-\infty)$ and $R(+\infty) = B(+\infty)$.
5. For \mathbb{R} , the useful function R is constant.

Stability Conditions

If we choose a more complicated continuous quiver of type A then the axioms become more complicated and there is also a 6th axiom.

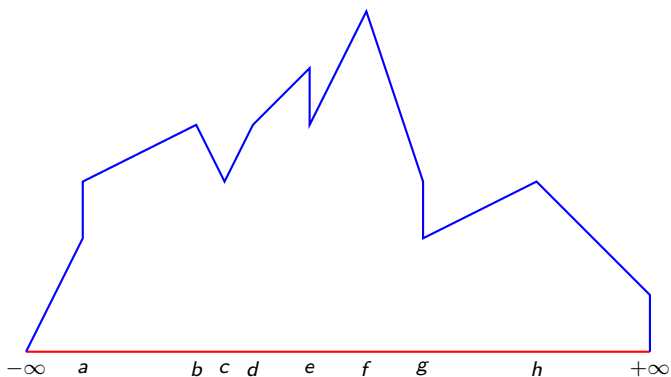
Definition

A **stability condition** for \mathbb{R} is an equivalence class σ of red-blue pairs, where two pairs (R, B) and (R', B') are equivalent if they are related by a vertical translation.

Stability Conditions

Example

Here is an example for our \mathbb{R} .



Stability Conditions

“Definition”

Given a stability condition σ , we say $M_{[a,b]}$ in $\text{rep}^{\text{fp}}(\mathbb{R})$ is **σ -semistable** if there is a horizontal line segment ℓ contained inside the graph of σ such that the following are satisfied.

1. The left endpoint of ℓ has horizontal coordinate a .
2. The right endpoint of ℓ has horizontal coordinate b .
3. The endpoints of ℓ are in the graph of B .
4. The graph of B does not transversally cross ℓ anywhere.

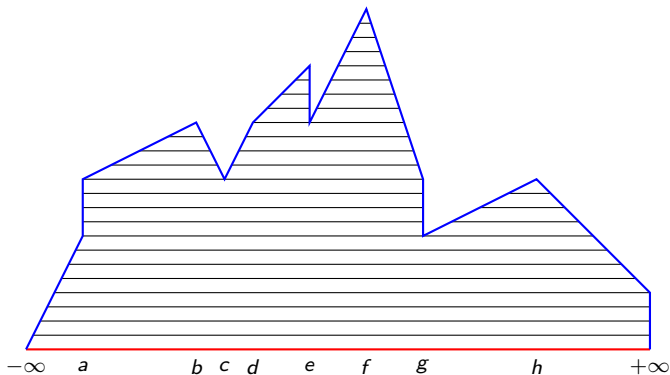
For (3) and (4), remember that we include the vertical lines connecting the function value, left limit, and right limit.

There may be more than one ℓ !

Stability Conditions

Example

The black horizontal lines each indicate a σ -semistable module.



Notice many choices of ℓ for $M_{[a,g]}$. We will come back to this.

Stability Conditions

Theorem (Igusa–R)

Via a process called **continuous tilting**, the stability conditions for any continuous quiver of type A are in bijection with the stability conditions for any other continuous quiver of type A .

Moreover, if the bijection corresponds σ and σ' , then the σ -semistable representations are in bijection with the σ' -semistable representations.

This is why we can just work with \mathbb{R} .

Stability Conditions

Definition

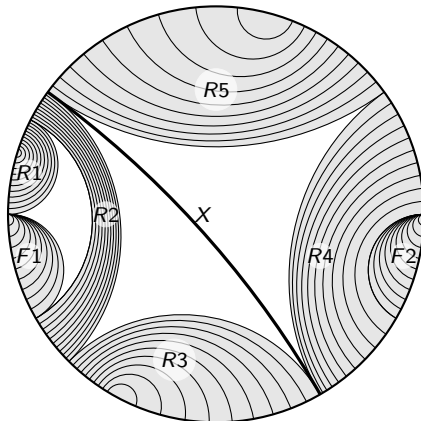
A **lamination** of the (Poincaré disk model of) the hyperbolic plane is a maximal collection of noncrossing ideal geodesics.

Fact

A lamination is a combination of **fountains**, **rainbows**, and **discrete geodesics**. (See next slide.)

Stability Conditions

Example



The F_n 's are fountains. The R_n 's are rainbows. The geodesic marked X is discrete.

Definition

A **measured lamination** is a pair $(\mathcal{L}, \mathcal{M})$, where \mathcal{L} is the lamination and \mathcal{M} is the measure on \mathcal{L} , satisfying the following conditions.

- ▶ The measure \mathcal{M} assigns to each maximal fountain, each maximal rainbow, and each discrete geodesic a positive real value such that the sum of all these converges in \mathbb{R} .
- ▶ Every fountain, rainbow, and discrete geodesic has positive measure.

Stability Conditions

Theorem (Igusa–R)

There is a bijection

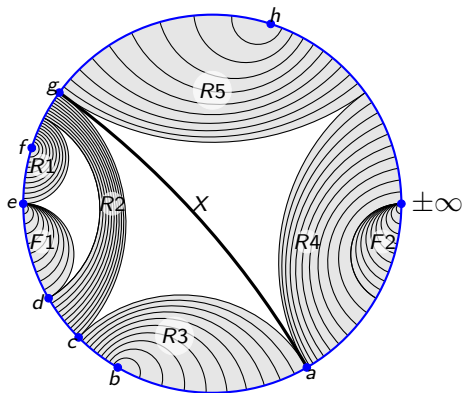
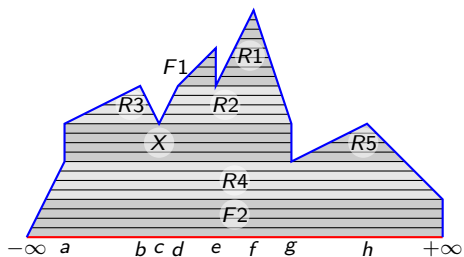
$$\{(\mathcal{L}, \mathcal{M})\} \longleftrightarrow \{\sigma \text{ satisfying the four point condition}\}.$$

The geodesics in a measured lamination $(\mathcal{L}, \mathcal{M})$ are in bijection with the σ -semistable representations of the corresponding stability condition σ .

The measure of a maximal fountain, maximal rainbow, or discrete geodesic is given by the height of the corresponding feature in the corresponding stability condition. (See next slide.)

Stability Conditions

Example



Stability Conditions

“Why did we start with representation theory?”

Theorem (Igusa–R)

Every continuous cluster of type A comes from a stability condition.

In the cluster category, the σ -semistable representations have the pairwise condition

$$\dim (\mathrm{Ext}(M, N) \oplus \mathrm{Ext}(N, M)) \leq 1.$$

(You don't need to understand this. Just know there's an algebraic motivation.)

The Plan

Stability Conditions

Nakayama Categories

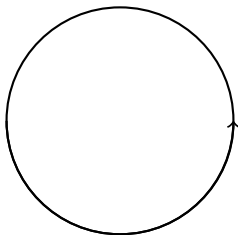
Preprojective Categories

Nakayama Categories

We keep the \mathbb{R} as before.

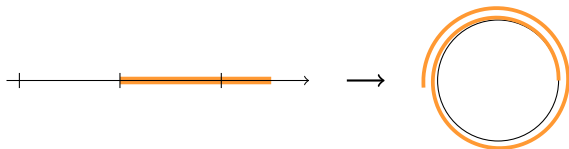
There is an action of \mathbb{Z} on \mathbb{R} (the category) by sending each x to $x + 1$ and each α_{yx} to $\alpha_{y+1,x+1}$.

If we take the orbit by this action, we obtain the circle category \mathbb{S}^1 .



Nakayama Categories

There is a functor, called the push-down functor, from $\text{rep}^{\text{pwf}}(\mathbb{R})$ to $\text{rep}^{\text{pwf}}(\mathbb{S}^1)$. It takes interval representations to **string representations**.



Theorem (Hanson–R)

Every pwf representation of \mathbb{S}^1 decomposes uniquely up to isomorphism into a direct sum of string representations (and finitely-many band representations).

Fact

Every string representation of \mathbb{S}^1 is a push down of an interval representation of \mathbb{R} .

Nakayama Categories

Definition

A **pre-Kupisch function** is a function $\kappa : I \rightarrow \mathbb{R}_{>0}$, for $I \subseteq \mathbb{R}$ such that:

- ▶ $K(t) := \kappa(t) + t$ is increasing on I and
- ▶ $[t, K(t)] \subset I$ for all $t \in I$.

We say κ is **Kupisch** if $\kappa(t+1) = \kappa(t)$ for all $t \in \mathbb{R}$.

An interval representation M_J of \mathbb{R} is compatible with a pre-Kupisch κ if $J \subset [t, K(t)]$ for some $t \in \mathbb{R}$.

A string representation \overline{M} of \mathbb{S}^1 is compatible with Kupisch κ if, for any interval representations M_J that pushes down to \overline{M} , we have M_J compatible with κ .

The pwf representations of \mathbb{R} (of \mathbb{S}^1) that are compatible with κ form an abelian category denoted $\text{rep}_{\kappa}^{\text{pwf}}(\mathbb{R})$ (denoted $\text{rep}_{\kappa}^{\text{pwf}}(\mathbb{S}^1)$). The category $\text{rep}_{\kappa}^{\text{pwf}}(\mathbb{S}^1)$ has no band representations.

Nakayama Categories

Definition

A **separation point** of a (pre-)Kupisch function κ is some $c \in \mathbb{R}$ such that the following hold:

1. $\lim_{t \rightarrow c^-} \kappa(t) = 0$ and
2. $K(t) < c$ for all $t < c$.

The set of separation points is denoted $\mathcal{S}(\kappa)$.

Notice that $c \in \mathcal{S}(\kappa)$ if and only if $c + 1 \in \mathcal{S}(\kappa)$.

Fact

Separation points are discontinuities of $K(t)$ of the first kind. The other kind of discontinuities are of the first kind (e.g., $K(t) = n + a$ for $n \leq t < \frac{1}{2}$ and $K(t) = n + \frac{1}{2} + a$ for $n + \frac{1}{2} \leq t < n + 1$, where $a > \frac{1}{2}$ in \mathbb{R}).

Notice that $K(t)$ is increasing.

Nakayama Categories

Theorem (Darboux—Froda)

A monotone increasing function has at most countably many discontinuities of the first kind.

Corollary (R–Zhu)

For a (pre-)Kupisch function κ , the function $K(t)$ has at most countably-many separation points.

“Definition”

An abelian category \mathcal{A} is **connected** if, for any categorical decomposition $\mathcal{A} \simeq \mathcal{A}_1 \oplus \mathcal{A}_2$, one of \mathcal{A}_1 or \mathcal{A}_2 is equivalent to \mathcal{A} and the other is the trivial category.

Theorem (R–Zhu)

- ▶ *The category $\text{rep}_\kappa^{\text{pwf}}(\mathbb{R})$ is connected if and only if $\mathcal{S}(\kappa) = \emptyset$.*
- ▶ *The category $\text{rep}_\kappa^{\text{pwf}}(\mathbb{S}^1)$ is connected if and only if $|\mathcal{S}(\kappa) \cap [0, 1)| \leq 1$.*

Theorem (R-Zhu)

- ▶ *Let $F : \text{rep}_\kappa^{\text{pwf}}(\mathbb{R}) \rightarrow \text{rep}_\lambda^{\text{pwf}}(\mathbb{R})$ be an equivalence of categories, for κ and λ pre-Kupsich functions on I and J respectively, such that $\mathcal{S}(\kappa) = \mathcal{S}(\lambda) = \emptyset$. Then F induces an orientation-preserving homeomorphism $f : I \rightarrow J$ such that $\lambda = f_*\kappa$, where $f_*\kappa$ is the push forward.*
- ▶ *Let $F : \text{rep}_\kappa^{\text{pwf}}(\mathbb{S}^1) \rightarrow \text{rep}_\lambda^{\text{pwf}}(\mathbb{S}^1)$ be an equivalence of categories, for κ and λ Kupisch functions, such that $\mathcal{S}(\kappa) = \mathcal{S}(\lambda) = \emptyset$. Then F induces an orientation-preserving homeomorphism $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\lambda = f_*\kappa$ (in the natural way).*

Nakayama Categories

Definition

A (topological) **dynamical system** is a pair (X, σ) where X is a topological space and $\sigma : X \rightarrow X$ is a continuous map.

Given two dynamical systems (X, σ) and (Y, τ) , we say a homeomorphism $f : X \rightarrow Y$ is a **conjugacy** if $f\sigma = \tau f$. If X and Y are both \mathbb{R} or \mathbb{S}^1 , we say f is **positive** if it is orientation-preserving.

A (pre-)Kupisch function κ naturally gives rise to a dynamical system (\mathbb{R}, K) or (\mathbb{S}^1, \bar{K}) , where \bar{K} is the induced map $\mathbb{S}^1 \rightarrow \mathbb{S}^1$.

Corollary (R-Zhu)

Classifying the categories $\text{rep}_{\kappa}^{\text{pwf}}(\mathbb{R})$ and $\text{rep}_{\kappa}^{\text{pwf}}(\mathbb{S}^1)$ is equivalent to classifying the corresponding dynamical systems up to positive conjugacy.

The Plan

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Nakayama Categories

Preprojective Categories

Preprojective Categories

Choose an open, bounded interval $\mathbb{I} \subset \mathbb{R}$.

We construct the category $\Lambda_{\mathbb{I}}$ from \mathbb{R} in the following way.

1. For each α_{yx} , we add $\alpha_{xy}^* : x \rightarrow y$ such that $\alpha_{xx}^* = \alpha_{xx} = \mathbf{1}_x$ and $\alpha_{xy}^* \alpha_{yz}^* = \alpha_{xz}^*$. (The α 's go right and the α^* 's go left.)
2. We add the relations

$$\alpha_{x+a-b, x+a}^* \alpha_{x+a, x} = \alpha_{x+a-b, x-b} \alpha_{x-b, x}^*$$

for every $x \in \mathbb{R}$ and $a, b \in \mathbb{R}_{>0}$.

3. Replace each Hom set with the \mathbb{k} -vector space whose basis is is every valid composition of α 's and α^* 's subject to the relations above and add a 0 object. (Make it additive \mathbb{k} -linear.)
4. Set every object outside of \mathbb{I} isomorphic to the 0 object.

In this presentation, we use $\mathbb{I} = (0, 1)$ to make things simpler.

Preprojective Categories

“Definition”

The category $\Lambda_{\mathbb{I}}$ has nonzero objects \mathbb{I} and morphisms are given by

$$\mathrm{Hom}_{\Lambda_{\mathbb{I}}}(x, y) = \mathbb{K}^{[|x-y|, \ell_{xy}]},$$

where $\ell_{xy} = \min\{x, y, 1-x, 1-y\}$. We think of the b th coordinate as a path of length b .

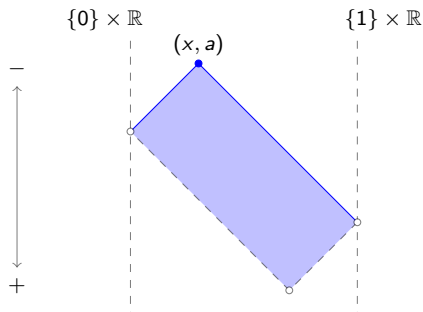
Given basis elements $f : x \rightarrow y$ and $g : y \rightarrow z$, the composition is gf , if the length of f plus the length of g is less than ℓ_{xz} , and 0 otherwise.

Pwf representations of $\Lambda_{\mathbb{I}}$ will not be helpful in this presentation.

Preprojective Categories

“Definition”

The **projective representation of $\Lambda_{\mathbb{I}}$ at x** is *drawn* like this:



where we choose a to be any real number. We denote the representation by P_x .

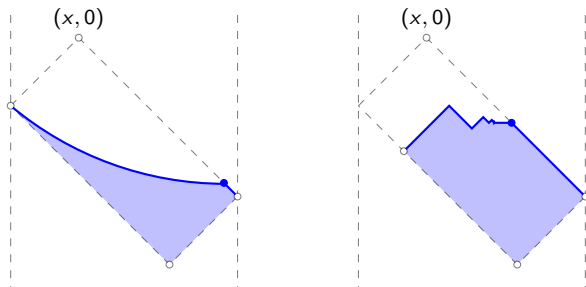
Notice the $+$ direction is *down*.

Preprojective Categories

Definition/Theorem (R-Thomas)

Draw P_x using $a = 0$. The possible **decorous**¹ **subrepresentations** of P_x are determined by the functions $\partial: [0, 1] \rightarrow \mathbb{R}$ such that $\partial(0) = x$, $\partial(1) = 1 - x$, and $|\partial(y) - \partial(z)| \leq |y - z|$, for all $y, z \in [0, 1]$.

Example



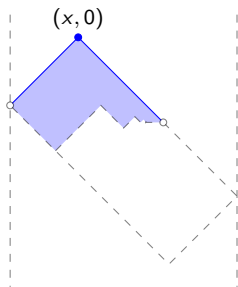
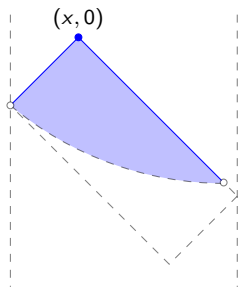
¹**decorous.** adj. “following the established traditions of refined society and good taste”

Preprojective Categories

Definition/Theorem (R-Thomas)

The **decorous quotient representations** are determined by the topological compliments inside the rectangle for P_x (and thus also determined by the same ∂ 's).

Example



Definition

A **permuton** is a measure μ on the σ -algebra of Borel sets of $[0, 1] \times [0, 1]$ such that $\mu([0, k] \times [0, 1]) = k = \mu([0, 1] \times [0, k])$.

You should think of this as a continuous generalization of a permutation matrix.

In fact...

Example

A permutation $w \in \mathfrak{S}_n$ defines a permuton γ_w as follows. Divide the unit square up into n equal rows and n equal columns, and then put a uniform measure of weight $1/n$ on each of the squares $(i, w(i))$.

Lemma (R-Thomas)

Let μ be a permuton. For fixed x , the function

$f_{\mu,x}(y) = -2\mu([0, y] \times [0, x]) + x + y$ has $f_{\mu,x}(0) = x$,
 $f_{\mu,x}(1) = 1 - x$, and $|f_{\mu,x}(y) - f_{\mu,x}(z)| \leq |y - z|$.

Thus, a permuton determines a decorous subrepresentation D_μ^x and decorous quotient representation U_μ^x for each $x \in (0, 1)$.

Theorem (R-Thomas)

Let μ be a permuton. Then, for any $x, y \in (0, 1)$, we have

$$\text{Hom}_{\text{Rep}(\Lambda_{\mathbb{I}})}(D_\mu^x, U_\mu^y) = 0.$$

Moreover, if we think of $\Lambda_{\mathbb{I}}$ as a rng (associative ring without unit), then $\bigoplus_{x \in (0,1)} D_\mu^x$ forms a two-sided ideal inside $\Lambda_{\mathbb{I}}$.

The End

Danke!