Decomposing Pointwise Finite-Dimensional Representations of the Circle

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1. Motivation

Representations of the circle have already appeared in several areas of mathematics.

- Burghelea and Dey studied them as circular persistence modules [1].
- Igusa and Todorov studied them to construct continuous Frobenius categories [5].
- Sala and Schiffmann studied them in the context of Fock spaces and the circle quantum group [6].
- Guillermou studied them in the context of sheaves, symplectic geometry, and cotangent bundles [3].

Why study these further?

- Each of the definitions used are somewhat different from the rest. A unified definition helps ou understanding of the general structure.
- Representations of $\mathbb{S}^1$ will help our understanding of representations of $\widetilde{\mathbb{A}}_n$ quivers.
- Path algebras of $\widetilde{\mathbb{A}}_n$ quivers are string algebras; the study of "continuous string algebras" remains largely unexplored territory.

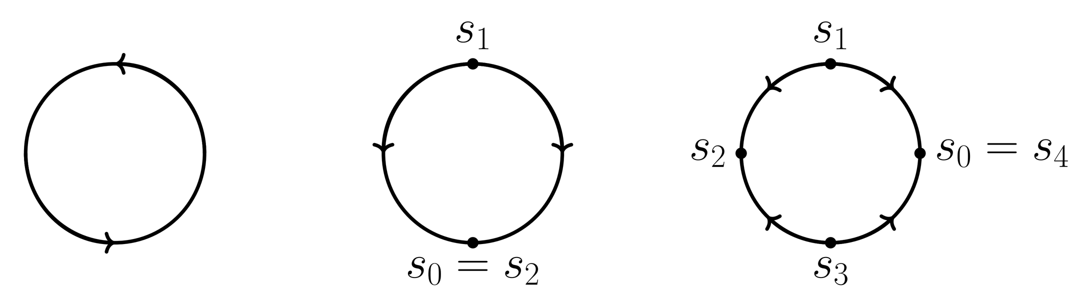
2. The Circle as a Category

Definition. Choose an even number of points S in $\mathbb{S}^1$ (possibly 0). If $S \neq \emptyset$, index the elements counterclockwise starting with s_0. Additionally, set $s_{|S|} = s_0$.

- If $S = \emptyset$ we give $\mathbb{S}^1$ the cyclic counterclockwise order.
- If $S \neq \emptyset$ the even-indexed elements of S are sinks and the odd-indexed elements are sources, inducing a partial order on $\mathbb{S}^1$.

We denote the order by $\preceq$ and call this our **orientation**.

Example.



$|S| = 0$ $|S| = 2$ $|S| = 4$

Definition. Given S and $\preceq$, **a continuous quiver of type** $\widetilde{\mathbb{A}}$ is a category Q.

- The objects of Q are the points of $\mathbb{S}^1$.
- If $|S| = 0$ then for each pair of distinct points $x, y \in \mathbb{S}^1$:

$$\mathrm{Hom}_Q(x,y) = \{g_{x,y} \circ \omega_x^n : n \in \mathbb{N}\} = \{\omega_y^n \circ g_{x,y} : n \in \mathbb{N}\}.$$

  - Here, $g_{x,y}$ is the unique morphism from x to y that travels counterclockwise less than one rotation.
  - And, ω_x (ω_y) is the unique map from x (y) to itself that travels around $\mathbb{S}^1$ once.
- If $|S| \geq 2$ then for each pair of distinct points $x, y \in \mathbb{S}^1$:

$$\mathrm{Hom}_Q(x,y) = \begin{cases} \{g^{\uparrow}_{x,y}, g^{\downarrow}_{x,y}\} & y = s_0, x = s_1, |S| = 2 \\ \{g_{x,y}\} & y \preceq x \\ \emptyset & y \not\preceq x. \end{cases}$$

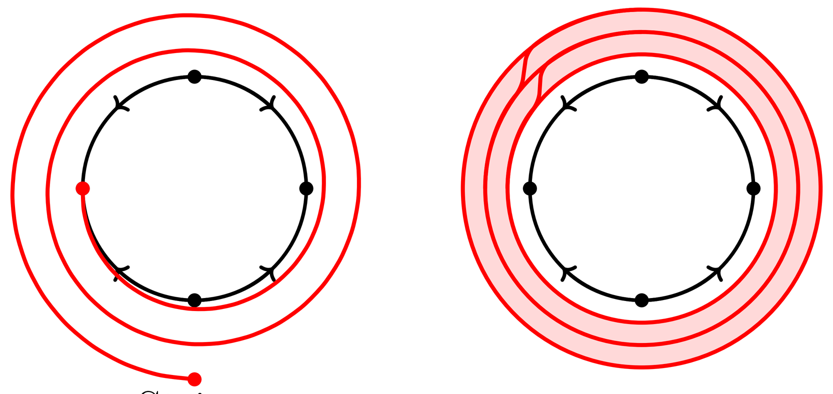
3. Representations of the Circle

Definition. Let Q be a continuous quiver of type $\widetilde{\mathbb{A}}$. A **representation** of Q is a functor V from Q to k-vector spaces. (We assume a field k has been fixed.)
If V factors through finite-dimensional k-vector spaces we say V is **pointwise finite-dimensional**. We will write this as **pwf** for short.

"Definition". Let V be representation of Q, a continuous quiver of type $\widetilde{\mathbb{A}}$. We call V a **string** if we can parameterize V by lifting to a bounded interval of $\mathbb{R}$.

"Definition". Let Q be a continuous quiver of type $\widetilde{\mathbb{A}}$ and V a representation of Q. If $V(g_{x,y})$ is an isomorphism for all $x, y \in \mathbb{S}^1$ and the map obtained by "traveling around" the circle cannot be written as a direct sum then V is a **band**.

Example.



String Band

Theorem (Hanson-R. [4])**.** *Let V and W be representations of a continuous quiver Q of type $\widetilde{\mathbb{A}}$.*

1. *Suppose V and W are strings. Then $V \cong W$ if and only if they lift to the same interval of $\mathbb{R}$ modulo 2π.*
2. *Suppose V and W are bands; let $\widehat{V}$ and $\widehat{W}$ be the "traveling around" maps for V and W, respectively. Then $V \cong W$ if and only if there is a matrix A such that $\widehat{V} = A^{-1}\widehat{W}A$.*
3. *If V is a string and W is a band then $V \ncong W$.*

4. Finitistic Representations

- If $|S| \geq 2$, for each $0 \leq i < |S|$ let $\overline{R}_i$ be the closed region on $\mathbb{S}^1$ from s_i to s_{i+1}. The interior of the region is denoted R_i.
- If $|S| = 0$ we just use the regions given by angles 0 to π and by angles π to 2π; call these regions $\overline{R}_0$ and $\overline{R}_1$, respectively. Then respective interiors are R_0 and R_1.

Lemma (Hanson-R. [4])**.** *Suppose W is a summand of V restricted to $\overline{R}_i$ and the support of W is contained in R_i. Then W is a summand of V.*

- Each $\overline{R}_i$ is totally-ordered.
- Crawley-Boevey proved that representations of totally-ordered sets decompose into interval indecomposables [2] (in our case, strings).
- Applied to each $\overline{R}_i$ we have $V \cong V' \oplus \left(\bigoplus_{i=0}^{|S|-1} W_i\right)$, where each W_i is the sum of summands of V whose support is contained in R_i.

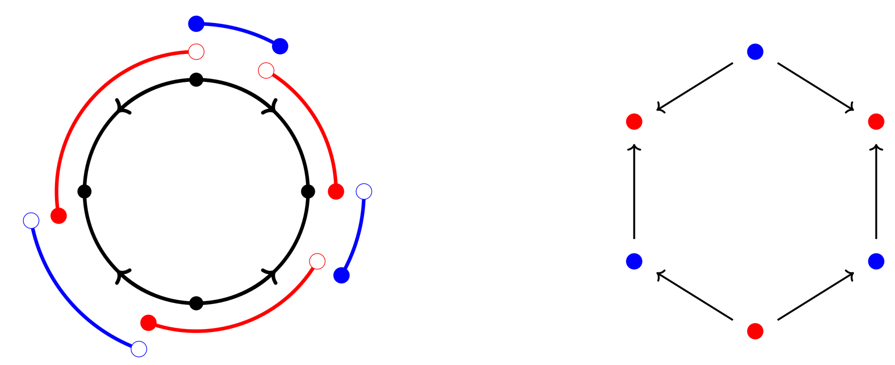
Definition. A representation V is called **finitistic** if each W_i as described is 0.

5. Decomposition

Lemma (Hanson-R. [4]) *A finitistic representation V partitions the continuous quiver Q into finitely-many pieces. On each piece V is constant up to isomorphism.*

Thus we may "lift" V to a representation M_V of a quiver Q_V.

Example.



Partition Q_V

We then examine the structure of M_V and "push it down."

Lemma (Hanson-R. [4])**.**

- *A string or band representation of Q_V "pushes down" to a respective string or band representation of Q.*
- *An indecomposable representation of Q_V "pushes down" to an indecomposable representation of Q.*
- *Direct sums commute with "pushing down".*

We now have enough to understand the summands of a pwf representation of $\mathbb{S}^1$.

Theorem (Hanson-R. [4])**.**

- *A pointwise finite-dimensional representation of Q decomposes into a direct sum of string and band representations.*
- *A pointwise finite-dimensional representation of Q is indecomposable if and only if it is either a string or band representation.*

References

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