Categories from generalized thread quivers

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1. The Basics

A ***thread quiver*** (generalizing Berg and van Roosmalen's construction) consists of a pair $(Q,\mathcal{P})$, where Q is a (possibly infinite) quiver and $\mathcal{P}$ is a collection of totally ordered sets $\{\mathcal{P}_\alpha\}_{\alpha\in Q_1}$ indexed by the arrows of Q. For each $\mathcal{P}_\alpha$, the set $\overline{\mathcal{P}}_\alpha$ is $\mathcal{P}_\alpha\cup\{\min\alpha,\max\alpha\}$, where $\min\alpha$ and $\max\alpha$ are the minimum and maximum elements, respectively. We have $|\overline{\mathcal{P}}_\alpha\setminus\mathcal{P}_\alpha|=2$.

A ***finite subthreading*** of $(Q,\mathcal{P})$ is a thread quiver $(Q,\mathcal{P}')$ such that each $\mathcal{P}'_\alpha$ is finite and $\mathcal{P}'_\alpha\subseteq\mathcal{P}_\alpha$ for each $\alpha\in Q_1$.

We "replace" each arrow α in Q with the totally ordered set $\mathcal{P}_\alpha$ and obtain a ($\Bbbk$-linear) category $\mathcal{C}$, called the ***path category***.

- The objects in $\mathcal{C}$ are $(\bigcup_{\alpha\in Q_1}\mathcal{P}_\alpha)\cup Q_0$.
- The morphisms in $\mathcal{C}$ are generated from path-like elements η_{yx} where $x\le y\in\overline{\mathcal{P}}_\alpha$ for some $\alpha\in Q_1$. If $x=y$ then $e_x:=\eta_{xx}=\mathbf{1}_x$.
- For composition: $\eta_{zy}\eta_{yx}=\eta_{zx}$, for $x\le y\le z$ in some $\overline{\mathcal{P}}_\alpha$. If $x=s(\alpha)$ and $z=t(\alpha)$ then we identify η_{zx} and α (unless $s(\alpha)=t(\alpha)$).

A ***pointwise finite-dimensional (pwf) representation*** of $\mathcal{C}$ is a functor $M:\mathcal{C}\to\operatorname{mod}(\Bbbk)$ and $\operatorname{rep}^{\mathrm{pwf}}\mathcal{C}$ is the functor category.

Let $\mathcal{I}$ be an ideal in $\mathcal{C}$ and $\mathcal{A}:=\mathcal{C}/\mathcal{I}$. We call $\mathcal{I}$ ***weakly admissible*** if $\mathcal{I}$ satisfies 3 conditions, including $\operatorname{End}_{\mathcal{A}}(x)$ is local for all $x\in\operatorname{Ob}(\mathcal{C})$.

We say a representation M is ***noise*** if it is entirely supported on some $\mathcal{P}_\alpha$. (Note this excludes Q_0!) A representation is ***noise free*** if none of its direct summands are noise. See the pictures for examples.

2. Decomposition

Item 1 predominantly follows from [1, 2].

Theorem A (Paquette–R–Yıldırım '24)**.** *Let $(Q,\mathcal{P})$ be a thread quiver, $\mathcal{C}$ be its path category, $\mathcal{I}$ be a weakly admissible ideal, and $\mathcal{A}=\mathcal{C}/\mathcal{I}$. Let M be in $\operatorname{rep}^{pwf}\mathcal{A}$.*

1. *$\operatorname{rep}^{pwf}\mathcal{A}$ has the Krull–Remak–Schmidt–Azumaya property.*
2. *Each indecomposable summand of M is either noise or noise free, in the noise case an interval indecomposable and in the noise free case coming from a finite subthreading of $(Q,\mathcal{P})$.*
3. *If Q is finite, only finitely-many indecomposable summands of M are noise free.*

3. Projective and injective objects

For each interval J of each $\overline{\mathcal{P}}_\alpha$ there are corresponding projective object P_J and injective object I_J in $\operatorname{rep}^{\mathrm{pwf}}\mathcal{A}$, both indecomposable. Note that such a J may not have a minimal (maximal) element and so P_J may not be representable (I_J may not be corepresentable). See the pictures for examples of such J.

Proposition (Paquette–R–Yıldırım '24)**.** *$P_J\cong P_{J'}$ if there exists $y\in J\cap J'$ such that, for all $x\le y$ in $\overline{\mathcal{P}}_\alpha$, we have $x\in J$ if and only if $x\in J'$. A dual statement holds for injective I_J's.*

We say $\mathcal{A}$ is ***left Q-bounded*** if, for each $x\in Q_0$, there are at most finitely-many $y\in Q_0$ such that $\operatorname{Hom}_{\mathcal{A}}(y,x)\neq 0$. The dual condition is ***right Q-bounded***. If $\mathcal{A}$ is both left and right Q-bounded then we just say it is ***Q-bounded***.

Theorem B (Paquette–R–Yıldırım '24)**.** *Let $(Q,\mathcal{P})$ be a thread quiver, $\mathcal{C}$ be its path category, $\mathcal{I}$ be a weakly admissible ideal, and $\mathcal{A}=\mathcal{C}/\mathcal{I}$. If $\mathcal{A}$ is left Q-bounded then every indecomposable projective object is isomorphic to P_J for some interval $J\subset\overline{\mathcal{P}}_\alpha$. Dually, if $\mathcal{A}$ is right Q-bounded then every indecomposable injective object is isomorphic to I_J for some interval $J\subset\overline{\mathcal{P}}_\alpha$.*

4. Hereditary-ness

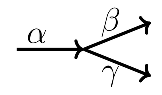
The subcategory $\operatorname{rep}^{\mathrm{qnf}}\mathcal{A}$ of $\operatorname{rep}^{\mathrm{pwf}}\mathcal{A}$ consists of pwf representations M of $\mathcal{A}$ such that for each $\alpha\in Q_1$ there are at most finitely many noise direct summands of M supported on $\mathcal{P}_\alpha$. The category $\operatorname{rep}^{\mathrm{qnf}}\mathcal{A}$ is a(n abelian) Serre subcategory of $\operatorname{rep}^{\mathrm{pwf}}\mathcal{A}$.

The subcategory $\operatorname{rep}^{\mathrm{fp}}\mathcal{A}$ of $\operatorname{rep}^{\mathrm{pwf}}\mathcal{A}$ consists of pwf representations finitely presented by representable projectives. The category $\operatorname{rep}^{\mathrm{fp}}\mathcal{A}$ is an abelian subcategory of $\operatorname{rep}^{\mathrm{pwf}}\mathcal{A}$ but it is not a Serre subcategory.

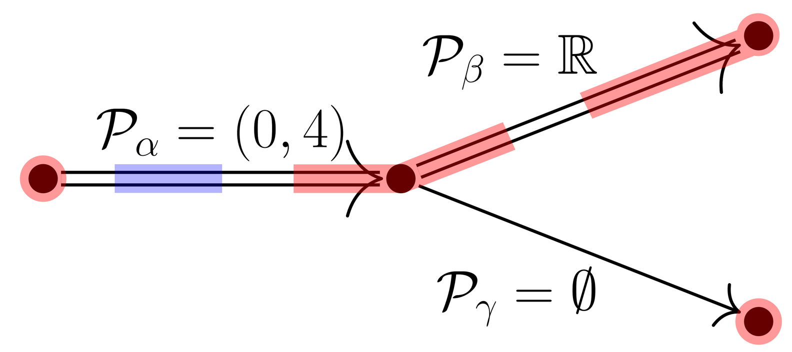
We have the following statements about the hereditary property.

Theorem C (Paquette–R–Yıldırım '24)**.** *Let $(Q,\mathcal{P})$ be a thread quiver and let $\mathcal{C}$ be its path category.*

1. *The category $\operatorname{rep}^{qnf}\mathcal{C}$ is hereditary.*
2. *If $\mathcal{C}$ is left Q-bounded then both $\operatorname{rep}^{pwf}\mathcal{A}$ and $\operatorname{rep}^{fp}\mathcal{A}$ are hereditary.*
3. *If $\mathcal{C}$ is right Q-bounded then $\operatorname{rep}^{pwf}\mathcal{A}$ is hereditary.*

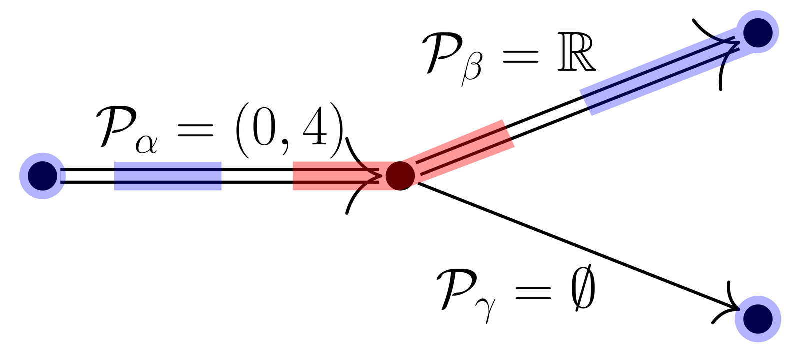
In these pictures, Q is  and $\mathcal{C}$ is the path category of $(Q,\mathcal{P})$ where $\mathcal{P}_\alpha=(0,4)$, $\mathcal{P}_\beta=\mathbb{R}$, and $\mathcal{P}_\gamma=\emptyset$. The representation categories $\operatorname{rep}^{\mathrm{pwf}}\mathcal{C}$, $\operatorname{rep}^{\mathrm{qnf}}\mathcal{C}$, and $\operatorname{rep}^{\mathrm{fp}}\mathcal{C}$ are all hereditary.

Support for noise indecomposables:



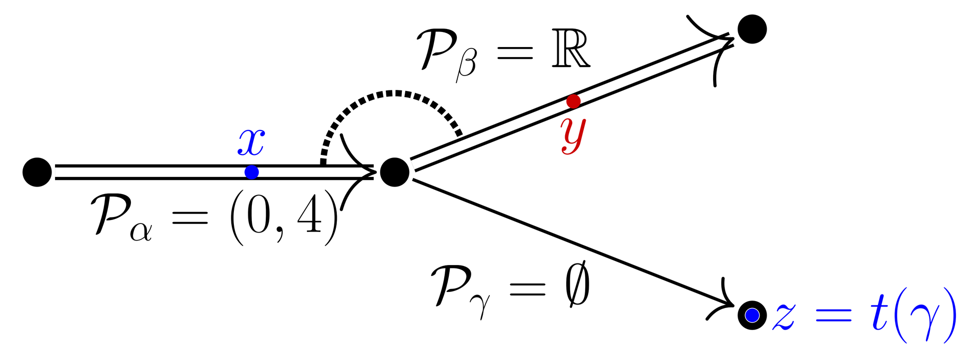
- This can be the support of a noise indecomposable.
- This cannot be the support of a noise indecomposable.

Possible projective/injectes intervals:



- This could be the J for P_J and I_J.
- This cannot be the J for any P_J or I_J.

Special biserial / gentle category:



Let $\mathcal{I}=\langle\{\text{paths } x\to y\mid x\in\mathcal{P}_\alpha, y\in\mathcal{P}_\beta\}\rangle$. Then, $\mathcal{A}=\mathcal{C}/\mathcal{I}$ behaves like a special biserial algebra, in particular a gentle algebra. In the diagram, we have a path from x to z but not from x to y.

References

[1] M. B. Botnan and W. Crawley-Boevey, *Decomposition of persistence modules*, Proc. Amer. Math. Soc. 148(11), 4581–4596, 2020, DOI: 10.1090/proc/14790.

[2] P. Gabriel and A. V. Roĭter, Representations of Finite-Dimensional Algebras Springer Berlin, Heidelberg, 1997.

Please see the paper for our complete list of references.