A Continuous Associahedron of Type A

Joint with M. C. Kulkarni, J. P. Matherne, and K. Mousavand

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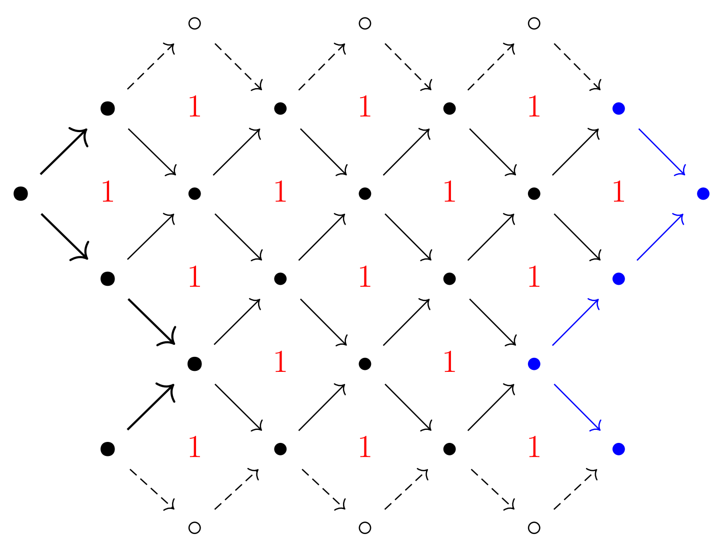
Motivation. The associahedron can be viewed as the "cluster polytope" which captures the combinatorics of type A cluster algebras [5, 4]. Representation theory of type A quivers with continuously-many vertices yields analogous cluster structures [8, 6, 7]. The continuous case does not yet have a related cluster algebra. We wish to strengthen the relationship between the finite and continuous cases with a continuous analogue of the cluster polytope.

The scattering amplitude of a certain quantum field theory is treated in [1]. Those authors realized the corresponding amplituhedron as the generalized associahedron of type A. A continuous version is constructed in [1] as an inverse limit of the finite constructions for type A_n, as n goes to infinity. We wish to construct our continuous associahedron from a representation theoretic point of view using a continuous cluster structure. This allows us to avoid taking the (inverse) limit of finite structures.

Finite Case. We show modified version of the technique used by the authors of [3] through an example. Fix a field $\Bbbk$. Consider the A_5 quiver $Q = [1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5]$ and its *augmented Auslander–Reiten quiver* (augmented AR quiver).

The black bullets are isomorphism classes of indecomposables in $\operatorname{rep}_{\Bbbk} Q$. The bold black bullets and bold arrows on the left make up the projective slice, we will call it $\mathcal{Z}$. The blue bullets and arrows are $\mathcal{Z}[1]$. Following [3] we assign a positive value $\underline{c}(M)$ to each indecomposable in $\operatorname{rep}_{\Bbbk} Q$. We consider functions Φ, from the indecomposables in $\operatorname{rep}_{\Bbbk} Q$ and $\mathcal{Z}[1]$ to $\mathbb{R}_{>0}$, such that, for every Auslander–Reiten triangle $X \to Y \oplus Z \to W \to$ in $\mathcal{D}^b(Q)$, we have

$$\Phi(X)+\Phi(W) = \Phi(Y)+\Phi(Z)+\underline{c}(X).$$



These equations are called the *deformed mesh relations*. (Think of the mesh relations being deformed by $\underline{c}$.) Notice that in the equations, Y or Z may be 0 since there are Auslander–Reiten triangles of this form in $\mathcal{D}^b(Q)$. The equations are building blocks to equations for any rectangle tilted at 45° and contained in the augmented Auslander–Reiten quiver.

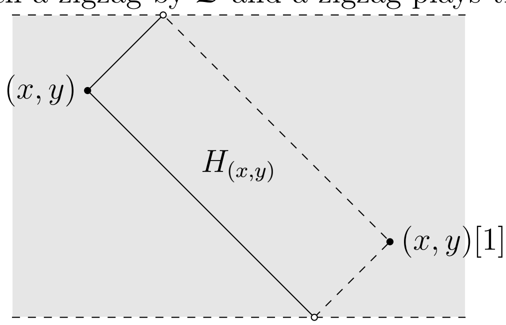
Consider two adjacent Auslander–Reiten triangles in the augmented AR quiver above: $X \to Y \oplus Z \to W \to$ and $Y \to W \oplus R \to T \to$. They yield a new distinguished triangle $X \to Z \oplus R \to T \to$ and a new equation $\Phi(X) + \Phi(T) = \Phi(Z) + \Phi(R) + \underline{c}(X) + \underline{c}(Y)$. We think of this by the following mantra: "the sides are equal to the top and bottom plus the sum of $\underline{c}$'s on the interior."

The space of all nonnegative Φ that satisfy the deformed mesh relations for a fixed $\underline{c}$ yield a 5-dimensional polytope in $\coprod R_{\operatorname{Ind}(\operatorname{rep}_{\Bbbk} Q)\sqcup \operatorname{Ind}(\mathcal{Z}[1])}$. This polytope encodes the same combinatorial structure as the A_5 associahedron, which in turn encodes the combinatorics of the A_5 cluster algebra.

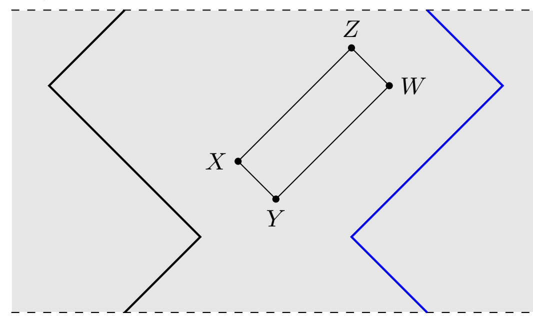
$\mathcal{D}$ and $\mathcal{Z}$. We first generalize the derived category $\mathcal{D}^b(Q)$ and the projective slice $\mathcal{Z}$. We define the $\Bbbk$-linear triangulated category $\mathcal{D}$ (equivalent to $\mathcal{D}_\pi$ in [8]) as follows. The indecomposable objects are given by $\mathrm{Ind}(\mathcal{D}) = \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$. We define shift of a point (x, y) by $(x, y)[1] = (x + \pi, -y)$.

Notice (x, y), $(x, y)[1]$, $(x + \frac{\pi}{2} - y, \frac{\pi}{2})$, and $(x + \frac{\pi}{2} + y, -\frac{\pi}{2})$ are the corners of a rectangle in $\mathbb{R}^2$. Let $H_{(x,y)}$ be the left sides and the interior of this rectangle. For two points $X = (x, y)$ and $Z = (z, w)$, we define $\mathrm{Hom}(X, Z)$ to be $\Bbbk$ if Z is in the H_X and 0 otherwise. Composition is given by multiplication in $\Bbbk$ if the composition is not 0.

We define a *zigzag* to be a set of line segments in $\mathcal{D}$ that each have slope ± 1 and form a zigzag shape from the top boundary of $\mathcal{D}$ to the bottom boundary of $\mathcal{D}$. We usually denote such a zigzag by $\mathcal{Z}$ and a zigzag plays the role of the projective slice from the finite case.



A point (x, y), it's shift $(x, y)[1]$, and $H_{(x,y)}$ (which includes the solid lines but not the dashed lines).



A zigzag $\mathcal{Z}$ and its shift $\mathcal{Z}[1]$ in blue, which bound $\mathrm{Ind}(\mathcal{C}_\mathcal{Z})$. Also depicted, a tilting rectangle $XYWZ$.

$\mathcal{C}_\mathcal{Z}$, tilting Rectangles, and $\underline{c}$. Consider a zigzag $\mathcal{Z}$ and its shift $\mathcal{Z}[1]$ (taken point-wise). We define the $\Bbbk$-linear category $\mathcal{C}_\mathcal{Z}$ to be the full subcategory of $\mathcal{D}$ whose indecomposables are those in $\mathcal{D}$ bounded by $\mathcal{Z}$ and $\mathcal{Z}[1]$ (inclusive) and by the boundary of $\mathcal{D}$. A *tilting rectangle* is a rectangle in $\mathbb{R}^2$ tilted at 45° whose interior is entirely contained in $\mathrm{Ind}(\mathcal{C}_\mathcal{Z})$. Notice this means the top or bottom corner may not be in $\mathrm{Ind}(\mathcal{C}_\mathcal{Z})$. Now, define a function $\underline{c} : \mathrm{Ind}(\mathcal{C}_\mathcal{Z}) \to \mathbb{R}_{>0}$. such that $\underline{c}$ is integrable (in the analytical sense) on $\mathrm{Ind}(\mathcal{C}_\mathcal{Z})$.

Construction and Results. Fix $\mathcal{C}_\mathcal{Z}$ and $\underline{c}$. Let $\Phi : \mathrm{Ind}(\mathcal{C}) \to \mathbb{R}_{\geq 0}$ be a function. We say Φ satisfies the *continuous deformed mesh relations* if, for every tilting rectangle $XYWZ$ in $\mathrm{Ind}(\mathcal{C}_\mathcal{Z})$ the following equation is satisfied: $\Phi(X) + \Phi(W) = \Phi(Y) + \Phi(Z) + \int_{XYZW} \underline{c}$. Notice how this is analogous to the finite-dimensional case. We define $\mathbb{U}_{\mathcal{Z},\underline{c}}$ to be the set of Φ as above satisfying the deformed mesh relations, called a *continuous associahedron of type A*.

Let X and W be in $\mathrm{Ind}(\mathcal{C}_\mathcal{Z})$. We say X and Y are $\mathbf{T}$-compatible if and only if there exists a tilting rectangle $XYWZ$ or $WYXZ$. This compatibility condition yields a *cluster theory* as defined in [7], which is similar to a cluster structure. A cluster theory comes with clusters and mutation; however we do not require that every element of a cluster be mutable.

Theorem. *Let $\mathbb{U}_{\mathcal{Z},\underline{c}}$ be a continuous associahedron of type A.*

(1) *$\mathbb{U}_{\mathcal{Z},\underline{c}}$ is convex in the sense that any line segment whose endpoints are contained in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is entirely contained in $\mathbb{U}_{\mathcal{Z},\underline{c}}$.*

(2) *If Φ satisfies the continuous deformed mesh relations and $\Phi(X) = 0$ for each X in a $\mathbf{T}$-cluster T, then Φ is on "boundary" of $\mathbb{U}_{\mathcal{Z},\underline{c}}$.*

(3) *Suppose $\mathcal{Z}$ has one line segment and denote by $\mathbb{U}_n$ the associahedron of type A_n. There is a sequence of embeddings $\mathbb{U}_2 \hookrightarrow \mathbb{U}_3 \hookrightarrow \cdots \hookrightarrow \mathbb{U}_n \hookrightarrow \mathbb{U}_{n+1} \hookrightarrow \cdots \mathbb{U}_{\mathcal{Z},\underline{c}}$. Embeddings $\mathbb{U}_n \hookrightarrow \mathbb{U}_{n+1}$ take clusters to clusters, respecting mutation. Embeddings $\mathbb{U}_n \hookrightarrow \mathbb{U}_{\mathcal{Z},\underline{c}}$ take clusters to $\mathbf{T}$-clusters and "take" mutations to mutations of $\mathbf{T}$-clusters.*

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