Cluster Theories and Examples in Type A

Motivation. Cluster categories were introduced in [BuanMarshReinekeReitenTodorov2006, CalderoChapatonSchiffler2006]. There are generalizations to the ∞-gon [HolmJørgensen2012] and completed ∞-gon [BaurGratz2018]. There is a further generalization to discrete laminations of the hyperbolic plane [IgusaTodorov2015b] and a further algebraic generalization in [IgusaR.Todorov2020]. Geometrically, all of these cluster structures should be related. However, functors cannot preserve all the triangulated and cluster structures simultaneously.

Cluster Theories and Abstract Cluster Structures. Cluster theories and embeddings of cluster theories were introduced in [IgusaR.Todorov2020]. The ability to mutate *every* object in a cluster is not required. A unique choice of mutation, if it exists, is required. Abstract cluster structures and the corresponding embeddings are introduced in [R.2020].

"Definition". Let $\mathcal{C}$ be a Krull–Schmidt category and $\mathbf{P}$ a pairwise compatibility condition on its (isoclasses of) indecomposables. *If it exists*, the **cluster theory** $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is a groupoid in the category of sets with embedding $I_{\mathbf{P},\mathcal{C}} : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathcal{S}et$. Objects of $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ are maximally $\mathbf{P}$-compatible sets of indecomposables in $\mathcal{C}$; morphisms are compositions of $\mathbf{P}$-mutations.

An **embedding of cluster theories** $(F, \eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ is an embedding $F : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ and a natural transformation $\eta : I_{\mathbf{P},\mathcal{C}} \to I_{\mathbf{Q},\mathcal{D}} \circ F$ whose component morphisms are injections.

"Definition". *If it exists*, the **abstract cluster structure** $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ is a subgroupoid of $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$. For each $\mathbf{P}$-cluster T, every $x \in T$ must be $\mathbf{P}$-mutable and $\mathcal{C}/\operatorname{add} T$ must be abelian. Furthermore, $\mathscr{S}_{\mathbf{P}}(\mathcal{C})$ must be maximal with respect to these properties.

Embeddings of abstract cluster structures are defined similarly to those for cluster theories. Every Dynkin type cluster category yields a cluster theory. However, the pairwise compatibility conditions in [HolmJørgensen2012, IgusaTodorov2015b], for example, yield cluster theories, not cluster structures. The cluster structures examined by those authors are indeed abstract cluster structures.

Main Results. Let $\mathbf{N}_m$ and $\mathbf{N}_n$ be the pairwise compatibility conditions for A_m and A_n, respectively, from [BuanMarshReinekeReitenTodorov2006, CalderoChapatonSchiffler2006]. Let $\mathbf{N}_\infty$, $\mathbf{N}_{\overline{\infty}}$, and $\mathbf{N}_{\mathbb{R}}$ be the pairwise compatibility conditions in [HolmJørgensen2012], [BaurGratz2018], and [IgusaTodorov2015b], respectively. Finally, let $\mathbf{E}$ be the pairwise compatibility condition introduced in [IgusaR.Todorov2020]. The cluster categories in the notation are suppressed in the theorems.

Theorem (R.2020)**.** *There is a chain of embeddings of cluster theories, for* $0 < m < n$*,*

$$\mathscr{T}_{\mathbf{N}_m} \longrightarrow \mathscr{T}_{\mathbf{N}_n} \longrightarrow \mathscr{T}_{\mathbf{N}_\infty} \longrightarrow \mathscr{T}_{\mathbf{N}_{\overline{\infty}}} \longrightarrow \mathscr{T}_{\mathbf{N}_{\mathbb{R}}} \longrightarrow \mathscr{T}_{\mathbf{E}}.$$

Theorem (R.2020)**.** *There is a commutative diagram of embeddings of cluster theories (left) that restricts to a commutative diagram of embeddings of abstract cluster structures (right):*

$$\begin{tikzcd}
 & \mathscr{T}_{\mathbf{N}_\infty} & \\
\mathscr{T}_{\mathbf{N}_m} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathscr{T}_{\mathbf{N}_n} \arrow[ul] \arrow[dl] \\
 & \mathscr{T}_{\mathbf{N}_{\mathbb{R}}} &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & \mathscr{S}_{\mathbf{N}_\infty} & \\
\mathscr{S}_{\mathbf{N}_m} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathscr{S}_{\mathbf{N}_n} \arrow[ul] \arrow[dl] \\
 & \mathscr{S}_{\mathbf{N}_{\mathbb{R}}}. &
\end{tikzcd}$$

Outlook. There are other type A cluster categories [IgusaTodorov2015a, PaquetteYıldırım2020]. Does including these cluster theories still yield a chain or something like a lattice instead? Do they have abstract cluster structures and if so how do they fit into the second theorem?

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